

## ON THE COMPLETENESS OF GRADIENT RICCI SOLITONS

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ABSTRACT. A gradient Ricci soliton is a triple  $(M, g, f)$  satisfying  $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$  for some real number  $\lambda$ . In this paper, we will show that the completeness of the metric  $g$  implies that of the vector field  $\nabla f$ .

## 1. INTRODUCTION

**Definition 1.1.** Let  $(M, g, X)$  be a smooth Riemannian manifold with  $X$  a smooth vector field. We call  $M$  a Ricci soliton if  $Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g$  for some real number  $\lambda$ . It is called shrinking when  $\lambda > 0$ , steady when  $\lambda = 0$ , and expanding when  $\lambda < 0$ . If  $(M, g, f)$  is a smooth Riemannian manifold where  $f$  is a smooth function, such that  $(M, g, \nabla f)$  is a Ricci soliton, i.e.  $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$ , we call  $(M, g, f)$  a gradient Ricci soliton and  $f$  the soliton function.

On the other hand, there has the following definition (see chapter 2 of [3]).

**Definition 1.2.** Let  $(M, g(t), X)$  be a smooth Riemannian manifold with a solution  $g(t)$  of the Ricci flow on a time interval  $(a, b)$  containing 0, where  $X$  is smooth vector field. We call  $(M, g(t), X)$  self-similar solution if there exist scalars  $\sigma(t)$  such that  $g(t) = \sigma(t)\varphi_t^*(g_0)$ , where the diffeomorphisms  $\varphi_t$  is generated by  $X$ . If the vector field  $X$  comes from a gradient of a smooth function  $f$ , then we call  $(M, g(t), f)$  a gradient self-similar solution.

It is easy to see that if  $(M, g(t), f)$  is a complete gradient self-similar solution, then  $(M, g(0), f)$  must be a complete gradient Ricci soliton. Conversely, when  $(M, g, f)$  is a complete gradient Ricci soliton and in addition, the vector field  $\nabla f$  is complete, it is well known (see for example Theorem 4.1 of [2]) that there is a complete gradient self-similar solution  $(M, g(t), f)$ ,  $t \in (a, b)$  (with  $0 \in (a, b)$ ), such that  $g(0) = g$ . Here we say that a vector field  $\nabla f$  is complete if it generates a family of diffeomorphisms  $\varphi_t$  of  $M$  for  $t \in (a, b)$ .

So when the vector field is complete, the definitions of gradient Ricci soliton and gradient self-similar solution are equivalent. In literature, people sometimes confuse the gradient Ricci solitons with the gradient self-similar solutions. Indeed, if the gradient Ricci soliton has bounded curvature, then it is not hard to see that the vector field  $\nabla f$  is complete. But, in general the soliton does not have bounded curvature.

The purpose of this paper is to show that the completeness of the metric  $g$  of a gradient Ricci soliton  $(M, g, f)$  implies that of the vector field  $\nabla f$ , even though the soliton does not have bounded curvature. Our main result is the following

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**Theorem 1.3.** *Let  $(M, g, f)$  be a gradient Ricci soliton. Suppose the metric  $g$  is complete, then we have:*

- (i)  $\nabla f$  is complete;
- (ii)  $R \geq 0$ , if the soliton is steady or shrinking;
- (iii)  $\exists C \geq 0$ , such that  $R \geq -C$ , if the soliton is expanding.

Indeed, we will show that the vector field  $\nabla f$  grows at most linearly and so it is integrable. Hence the above Definition 1.1 and 1.2 are equivalent when the metric is complete.

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## 2. GRADIENT RICCI SOLITONS

Let  $(M, g, f)$  be a gradient Ricci soliton, i.e.,  $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$ . By using the contracted second Bianchi identity we get the equation  $R + |\nabla f|^2 - 2\lambda f = \text{const.}$

**Definition 2.1.** Let  $(M, g, f)$  be a gradient shrinking or expanding soliton. By rescaling  $g$  and changing  $f$  by a constant we can assume  $\lambda \in \{-\frac{1}{2}, \frac{1}{2}\}$  and  $R + |\nabla f|^2 - 2\lambda f = 0$ . We call such a soliton normalized, and  $f$  a normalized soliton function.

**Proposition 2.2.** *Let  $(M, g, f)$  be a gradient Ricci soliton. Fix  $p$  on  $M$ , and define  $d(x) \triangleq d(p, x)$ , then the following hold*

- (i)  $\Delta R = \langle \nabla f, \nabla R \rangle + 2\lambda R - |\text{Ric}|^2$ ;
- (ii) Suppose  $\text{Ric} \leq (n-1)K$  on  $B_{r_0}(p)$ , for some positive numbers  $r_0$  and  $K$ . Then for arbitrary point  $x$ , outside  $B_{r_0}(p)$ , we have

$$\Delta d - \langle \nabla f, \nabla d \rangle \leq -\lambda d(x) + (n-1) \left\{ \frac{2}{3} K r_0 + r_0^{-1} \right\} + |\nabla f|(p).$$

*Proof.* (i) By using the soliton equation and the contracted second Bianchi identity  $\nabla_i R = 2g^{jk} \nabla_j R_{ik}$ , we have

$$\begin{aligned} \Delta R &= g^{ij} \nabla_i \nabla_j R = g^{ij} \nabla_i (2g^{kl} R_{jk} \nabla_l f) = 2g^{ij} g^{kl} \nabla_i (R_{jk} \nabla_l f) \\ &= 2g^{ij} g^{kl} \nabla_i (R_{jk}) \nabla_l f + 2g^{ij} g^{kl} R_{jk} \nabla_i \nabla_l f \\ &= g^{kl} \nabla_k R \nabla_l f + 2g^{ij} g^{kl} R_{jk} (\lambda g_{il} - R_{il}) \\ &= \langle \nabla f, \nabla R \rangle + 2\lambda R - 2|\text{Ric}|^2. \end{aligned}$$

(ii) Let  $\gamma : [0, d(x)] \rightarrow M$  be a shortest normal geodesic from  $p$  to  $x$ . We may assume that  $x$  and  $p$  are not conjugate to each other, otherwise we can understand the differential inequality in the barrier sense. Let  $\{\dot{\gamma}(0), e_1, \dots, e_{n-1}\}$  be an orthonormal basis of  $T_p M$ . Extend this basis parallel along  $\gamma$  to form a parallel orthonormal basis  $\{\dot{\gamma}(t), e_1(t), \dots, e_{n-1}(t)\}$  along  $\gamma$ .

Let  $X_i(t)$ ,  $i = 1, 2, \dots, n-1$ , be the Jacobian fields along  $\gamma$  with  $X_i(0) = 0$  and  $X_i(d(x)) = e_i(d(x))$ . Then it is well-known that (see for example [4])

$$\Delta d(x) = \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot{X}_i|^2 - R(\dot{\gamma}, X_i, \dot{\gamma}, X_i)] dt.$$

Define vector fields  $Y_i$ ,  $i = 1, 2, \dots, n-1$ , along  $\gamma$  as follows

$$Y_i(t) = \begin{cases} \frac{t}{r_0} e_i(t), & \text{if } t \in [0, r_0]; \\ e_i(t), & \text{if } t \in [r_0, d(x)]. \end{cases}$$

Then by using the standard index comparison theorem we have

$$\begin{aligned} \Delta d(x) &= \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot{X}_i|^2 - R(\dot{\gamma}, X_i, \dot{\gamma}, X_i)] dt \\ &\leq \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot{Y}_i|^2 - R(\dot{\gamma}, Y_i, \dot{\gamma}, Y_i)] dt \\ &= \int_0^{r_0} [\frac{n-1}{r_0^2} - \frac{t^2}{r_0^2} Ric(\dot{\gamma}, \dot{\gamma})] dt + \int_{r_0}^{d(x)} [-Ric(\dot{\gamma}, \dot{\gamma})] dt \\ &= - \int_0^{d(x)} Ric(\dot{\gamma}, \dot{\gamma}) dt + \int_0^{r_0} [\frac{n-1}{r_0^2} + (1 - \frac{t^2}{r_0^2}) Ric(\dot{\gamma}, \dot{\gamma})] dt \\ &\leq - \int_{\gamma} Ric(\dot{\gamma}, \dot{\gamma}) dt + (n-1) \left\{ \frac{2}{3} K r_0 + r_0^{-1} \right\}. \end{aligned}$$

On the other hand,

$$\langle \nabla f, \nabla d \rangle(x) = \nabla_{\dot{\gamma}} f(x) = \int_0^{d(x)} \left( \frac{d}{dt} \nabla_{\dot{\gamma}} f \right) dt + \nabla_{\dot{\gamma}} f(p) \geq \int_{\gamma} (\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} f) dt - |\nabla f|(p).$$

Using the soliton equation, we have

$$\begin{aligned} \Delta d - \langle \nabla f, \nabla d \rangle &\leq - \int_{\gamma} [Ric(\dot{\gamma}, \dot{\gamma}) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} f] dt + (n-1) \left\{ \frac{2}{3} K r_0 + r_0^{-1} \right\} + |\nabla f|(p) \\ &= -\lambda d(x) + (n-1) \left\{ \frac{2}{3} K r_0 + r_0^{-1} \right\} + |\nabla f|(p). \end{aligned}$$

□

Now we are ready to prove the theorem 1.3 .

*Proof.* Fix a point  $p$  on  $M$ , and define  $d(x) \triangleq d(p, x)$ . We divide the argument into three steps.

**Step 1** We want to prove a curvature estimate in the following assertion.

**Claim** *For any gradient Ricci soliton, we have:*

- (i) *If the soliton is shrinking or steady, then  $R \geq 0$ ;*
- (ii) *If the soliton is expanding, then there exist a nonnegative constant  $C = C(n)$  such that  $R \geq \lambda C$ .*

We only prove the case (i),  $\lambda \geq 0$ . Note that there is a positive constant  $r_0$ , such that  $Ric \leq (n-1)r_0^{-2}$  on  $B_{r_0}(p)$ , and  $|\nabla f|(p) \leq (n-1)r_0^{-1}$ , then by Proposition 2.2, we have

$$\Delta d - \langle \nabla f, \nabla d \rangle \leq \frac{8}{3} (n-1) r_0^{-1},$$

for any  $x \notin B_{r_0}(p)$ .

For any fixed constant  $A > 2$ , we consider the function  $u(x) = \varphi(\frac{d(x)}{A r_0}) R(x)$ , where  $\varphi$  is a fixed smooth nonnegative decreasing function such that  $\varphi = 1$  on  $(-\infty, \frac{1}{2}]$ , and  $\varphi = 0$  on  $[1, \infty)$ .

Then by Proposition 2.2, we have

$$\begin{aligned}\Delta u &= R\Delta\varphi + \varphi\Delta R + 2\langle \nabla\varphi, \nabla R \rangle \\ &= R(\varphi'' \frac{1}{(Ar_0)^2} + \varphi' \frac{1}{Ar_0} \Delta d) + \varphi(\langle \nabla f, \nabla R \rangle + 2\lambda R - |Ric|^2) + 2\langle \nabla\varphi, \nabla R \rangle.\end{aligned}$$

If  $\min_{x \in M} u \geq 0$ , then  $R \geq 0$  on  $B_{\frac{1}{2}Ar_0}(p)$ . Otherwise,  $\min_{x \in M} u < 0$ , then there exist some point  $x_1 \in B_{Ar_0}(p)$ , such that  $u(x_1) = \varphi R(x_1) = \min_{x \in M} u < 0$ . Because  $u(x_1)$  is the minimum of the function  $u(x)$ , we have  $\varphi' R(x_1) > 0$ ,  $\nabla u(x_1) = 0$ , and  $\Delta u(x_1) \geq 0$ .

Let us first consider the case that  $x_1 \notin B_{r_0}(p)$ . Then by direct computation, we have

$$\begin{aligned}\Delta u(x_1) &= (\frac{\varphi''}{\varphi} \frac{1}{(Ar_0)^2} + \frac{\varphi'}{\varphi} \frac{1}{Ar_0} \Delta d)u(x_1) - \frac{\varphi'}{\varphi} \frac{1}{Ar_0} \langle \nabla f, \nabla d \rangle u(x_1) \\ &\quad + 2\lambda u(x_1) - \varphi |Ric|^2 - \frac{\varphi'^2}{\varphi^2} \frac{2}{(Ar_0)^2} u(x_1) \\ &\leq (\frac{\varphi''}{\varphi} \frac{1}{(Ar_0)^2} - \frac{\varphi'^2}{\varphi^2} \frac{2}{(Ar_0)^2})u(x_1) - \frac{2}{n} \varphi R^2 \\ &\quad + \frac{\varphi'}{\varphi} \frac{1}{Ar_0} u(x_1) (\Delta d - \langle \nabla f, \nabla d \rangle). \\ &\leq (\frac{\varphi''}{\varphi} \frac{1}{(Ar_0)^2} - \frac{\varphi'^2}{\varphi^2} \frac{2}{(Ar_0)^2})u(x_1) - \frac{2}{n} \frac{1}{\varphi} u(x_1)^2 \\ &\quad + \frac{8}{3} (n-1) \frac{\varphi'}{\varphi} \frac{1}{Ar_0^2} u(x_1) \\ &= \frac{u(x_1)}{\varphi} \left\{ (\varphi'' \frac{1}{(Ar_0)^2} - \frac{\varphi'^2}{\varphi} \frac{2}{(Ar_0)^2}) + \frac{8}{3} (n-1) \varphi' \frac{1}{Ar_0^2} - \frac{2}{n} u(x_1) \right\} \\ &\leq \frac{|u(x_1)|}{\varphi} \left\{ \frac{\varphi'^2}{\varphi} \frac{2}{Ar_0^2} + \frac{8(n-1)}{3} (-\varphi') \frac{1}{Ar_0^2} + |\varphi''| \frac{1}{Ar_0^2} - \frac{2}{n} |u(x_1)| \right\}.\end{aligned}$$

Note that there exist a constant  $\tilde{C} = \tilde{C}(\varphi)$ , such that  $|\varphi'| \leq \tilde{C}$ ,  $\frac{\varphi'^2}{\varphi} \leq \tilde{C}$ , and  $|\varphi''| \leq \tilde{C}$ . So

$$|u(x_1)| \leq \frac{C}{Ar_0^2},$$

where the constant  $C = C(\varphi, n)$ , i.e.,  $R \geq -\frac{C}{Ar_0^2}$  on  $B_{\frac{1}{2}Ar_0}(p)$ .

We now consider the remaining case that  $x_1 \in B_{r_0}(p)$ . Then  $\varphi'(x_1) = \varphi''(x_1) = 0$ , and we have

$$\Delta u(x_1) = 2\lambda u(x_1) - \varphi |Ric|^2 \leq |u(x_1)| [-2\lambda - \frac{2}{n} |u(x_1)|].$$

Since  $\lambda \geq 0$ , we have  $|u(x_1)| \leq 0$ , i.e.,  $u(x_1) = 0$ . This is a contradiction.

Combining the above two cases, we have  $R \geq -\frac{C}{Ar_0^2}$  on  $B_{\frac{1}{2}Ar_0}(p)$  for any  $A > 2$ , which implies that  $R \geq 0$  on  $M$ .

The proof of (ii) is similar.

**Step 2** We next want to show that the gradient field grows at most linearly.

**Claim** *For any gradient Ricci soliton, there exist constants  $a$  and  $b$  depending only on the soliton, such that*

- (i)  $|\nabla f|(x) \leq |\lambda|d(x) + a$ ;
- (ii)  $|f|(x) \leq \frac{|\lambda|}{2}d(x)^2 + ad(x) + b$ .

For any point  $x$  on  $M$ , we connect  $p$  and  $x$  by a shortest normal geodesic  $\gamma(t), t \in [0, d(x)]$ .

We first consider that the soliton is steady, then  $R \geq 0$  and  $R + |\nabla f|^2 = C \geq 0$ , so we have  $|\nabla f| \leq \sqrt{C}$ .

Secondly, We consider that the soliton is shrinking. Without loss of generality, we may assume the soliton is normalized. So  $R \geq 0$  and  $R + |\nabla f|^2 - f = 0$ , these imply  $f \geq |\nabla f|^2$ . Let  $h(t) = f(\gamma(t))$ , then

$$|h'(t)| = |\langle \nabla f, \dot{\gamma} \rangle|(t) \leq |\nabla f|(\gamma(t)) \leq \sqrt{f(\gamma(t))} = \sqrt{h(t)}.$$

By integrating above inequality, we get  $|\sqrt{h(d(x))} - \sqrt{h(0)}| \leq \frac{1}{2}d(x)$ . Thus  $|\nabla f|(x) \leq \frac{1}{2}d(x) + \sqrt{f(p)}$ .

Finally, we consider that the soliton is expanding. Similarly we only need to show the normalized case. So  $R \geq -\frac{C}{2}$  and  $R + |\nabla f|^2 + f = 0$ , we obtain  $-f + \frac{C}{2} \geq |\nabla f|^2$ . Let  $h(t) = -f(\gamma(t)) + \frac{C}{2}$ , thus

$$|h'(t)| = |\langle \nabla f, \dot{\gamma} \rangle|(t) \leq |\nabla f|(\gamma(t)) \leq \sqrt{h(t)}.$$

By integrating above inequality, we get  $|\sqrt{h(d(x))} - \sqrt{h(0)}| \leq \frac{1}{2}d(x)$ . Thus  $|\nabla f|(x) \leq \frac{1}{2}d(x) + \sqrt{-f(p) + \frac{C}{2}}$ .

Therefore we have proved (i).

The conclusion (ii) follows from (i) immediately.

**Step 3** Since the gradient field  $\nabla f$  grows at most linearly, it must be integrable. Thus we have proved theorem 1.3 .  $\square$

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